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The viscous-diffusion nonlinear critical layer in a stratified shear flow

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Stationary finite-amplitude wave disturbances in a stratified shear flow with Richardson number larger than $\frac{1}{4}$ are investigated for large Reynolds numbers when viscosity and thermal conductivity, as well as nonlinearity, are essential factors in the critical layer. The jumps across the critical layer in average vorticity, reflection and transmission coefficients are calculated as functions of the local Reynolds number determined by the amplitude of the incident wave. With the increase of the incident wave amplitude the asymptotic value of the Richardson number on the same side of critical layer as the incident wave tends to $\frac{1}{4}$, the reflection coefficient tends to unity and the transmission coefficient to zero.

1. Introduction

The most interesting effects of wave–flow interactions occur in the vicinity of the critical layer (CL), where the phase velocity of the wave coincides with the flow velocity. Waves in large-Reynolds-number stratified flows are described within the linear non-dissipative approximation by the Taylor–Goldstein equation. This equation is known to have singularities in the CL. To remove them some additional factors should be taken into consideration, namely dissipation (Koppel 1964; Hazel 1967; Baldwin & Roberts 1970; Bowman, Thomas & Thomas 1980; van Duin & Kelder 1986); nonlinearity (Kelley & Maslowe 1970; Maslowe 1972, 1973, see also Maslowe 1986 and Stewartson 1981 and references therein); or non-stationarity (Booker & Bretherton 1967; Miles 1961; Howard 1961). In all the above-mentioned papers these factors are taken into account only very close to the CL, i.e. when the inner solution is constructed. Away from the CL vicinity the waves are considered to be linear, non-dissipative and stationary (outer solution), and the inner solution is used to connect the fields above and below the CL, i.e. to obtain rules for crossing the CL. On this point they principally differ from the works by Brown & Stewartson (1978), Brown, Rosen & Maslowe (1981), and Churilov & Shukhman (1987, 1988), where nonlinearity is taken into account both inside and outside the CL. However, the results are obtained in a weak-nonlinearity approximation by the method of small disturbances.

Non-stationary and dissipative theories are well known to yield the same transitional relations. The phase shift of the complex logarithmic function in the solution of the Taylor–Goldstein equation equals $-\pi$. Because of that, in the stably stratified case with Richardson number $Ri > \frac{1}{4}$ a wave passing through the CL is attenuated by the factor $e^{\pi\mu}$, where $\mu = (Ri - 0.25)^{\frac{1}{2}}$.

Neglect of dissipation gives a different picture of wave fields in the CL vicinity (Kelly & Maslowe 1970; Maslowe 1972, 1973). The patterns of streamlines are symmetrical about the CL, so the wave-field amplitudes are unchanged when the wave field passes through the CL and the phase shift equals zero.

In the works mentioned, the relations were obtained taking dissipation, nonlinearity and non-stationarity in the CL into account separately. Then the combined effect of the factors was considered. Brown & Stewartson (1980, 1982*a*, *b*) investigated wave–flow interaction for the case of a nonlinear, non-stationary, non-viscous CL at $Ri \gg 1$ in the weak nonlinear approximation. The time period for which these results are applicable is limited to while dissipation can be neglected. The results are valid when $t \ll H_{CL}^2/\nu$ (here H_{CL} is the CL thickness, and ν is the viscosity coefficient). On the other hand, when $t \gg H_{CL}^2/\nu$ time-dependence disappears the fields in the CL vicinity are determined by competition between dissipation and nonlinearity. The steady problem of the combined effect of dissipation and nonlinearity was considered by Haberman in 1972 for the homogeneous case, and in 1973 for the case of weak stratification ($Ri \gg 1$). The main purpose of the present work is to obtain rules for crossing the stationary CL that depend on the dissipation–nonlinearity relationship in its vicinity (i.e. on the internal vertical Reynolds number of the CL) for the case of a stratified shear flow with the Richardson number $Ri > \frac{1}{4}$. Since the approach to the problem is similar to that of Haberman (1972, 1973) the present paper follows his logics when describing the results.

The study has the following structure. The problem is formulated in §2. In §3 the system of equations describing the fields in the CL vicinity is presented and expressions connecting parameters of the mean flow and the wave amplitudes above and below the CL are obtained by considering the momentum and mass fluxes. In §4 the method for numerical integration of the system from §3 is presented. In §5 the results of numerical calculations are discussed.

2. Formulation of the problem

Consider a two-dimensional flow of stratified incompressible fluid taking into account its viscosity and thermal conductivity. Within the Boussinesq approximation the non-dimensional equations for vorticity and density have the following form:

$$\left.\begin{aligned} \frac{\partial}{\partial t}\nabla^2\Psi + \mathrm{J}(\nabla^2\Psi, \Psi) - \frac{\partial b}{\partial x} &= \frac{1}{Re}\nabla^2\nabla^2\Psi, \\ \frac{\partial}{\partial t}b + \mathrm{J}(b, \Psi) &= \frac{1}{Re\,Pr}\nabla^2 b. \end{aligned}\right\} \tag{2.1}$$

Here $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator, $\mathrm{J}(a,b) = \partial(a,b)/\partial(x,y)$ is a Jacobian, $x, y = x_{dim}/L_0, y_{dim}/L_0$ are non-dimensional horizontal and vertical coordinates, L_0 is the scale of flow, $t = t_{dim}U_0/L_0$ non-dimensional time, U_0 the scale of velocity, $\Psi = \Psi_{dim}/(U_0 L_0)$ the non-dimensional stream function,

$$b = (\rho - \rho_0)g/(N_0^2 L_0 \rho_0)$$

non-dimensional density, N_0 the characteristic value of the Brunt–Väisälä frequency, $Re = U_0 L_0/\nu$ the Reynolds number defined in terms of the parameters of the background flow; $Pr = \nu/\nu_t$ the Prandtl number (for water $Pr = 7$, for air $Pr = 0.7$–1), and ν, ν_t are viscosity and thermal conductivity coefficients.

For $Re_0 \gg 1$ the solution of (2.1) for a region far from the CL and the boundary layers can be found as a series expansion in Re_0^{-1}:

$$(\psi_*, b_*) = (\psi, b) + Re_0^{-1}(\psi_1, b_1) + \dots,$$

where (ψ, b) satisfy system (2.1) and viscosity and heat conductivity are neglected.

Consider a plane harmonic wave disturbance of small amplitude ϵ against a background flow with velocity profile $U_0(y)$, and density profile $b_0(y)$. Then it is possible to find a solution of (2.1) in the form of a series expansion in ϵ. Therefore

$$\left.\begin{aligned} \Psi &= \int U_0(y)\,\mathrm{d}y + \epsilon \sum_{n=1}^{\infty} \mathrm{Re}\,[\Psi_n^{(1)}(y) \exp\{\mathrm{i}n(\omega t - kx)\}] + \epsilon^2 \Psi^{(2)} + \dots, \\ b &= \int b_0(y)\,\mathrm{d}y + \epsilon \sum_{n=1}^{\infty} \mathrm{Re}\,[b_n^{(1)}(y) \exp\{\mathrm{i}n(\omega t - kx)\}] + \epsilon^2 b^{(2)} + \dots. \end{aligned}\right\} \qquad (2.2)$$

Here ω, k are the frequency and the wavenumber, respectively.

It should be mentioned that both the fundamental harmonic and higher harmonics arising from the nonlinear wave–flow interaction in the CL vicinity occur far from the CL. The high harmonics appearing from the wave incidence on the CL have been found by Brown & Stewartson (1982*b*). In that paper, however, only the first stage of the flow formation in the CL vicinity was studied when $t \ll Re_0 \epsilon^{\frac{4}{3}}$ (giving an expression equivalent to that in §1; see also the definition of the CL thickness in §3), and the influence of dissipation could be neglected. In the present paper the stage of flow stabilization when $t \gg Re_0 \epsilon^{\frac{4}{3}}$ is considered.

The functions $\psi_n^{(1)}(y)$, $b_n^{(1)}(y)$ can be determined from (2.1) to the first order of ϵ and zeroth order of Re_0^{-1}. The stream-function disturbance appears to satisfy the Taylor–Goldstein equation:

$$\frac{\mathrm{d}^2 \Psi_n^{(1)}}{\mathrm{d}y^2} + \frac{U_{0yy}}{c - U_0} \Psi_n^{(1)} + \left(\frac{N^2}{(c-U_0)^2} - n^2 k^2 \right) \Psi_n^{(1)} = 0 \quad \text{for} \quad n = 1, \dots \qquad (2.3)$$

and the normalized density disturbance b_n is connected with Ψ_n by

$$b_n^{(1)} = -N^2 \Psi_n^{(1)} (U_0 - c)^{-1}. \qquad (2.4)$$

Here $N^2 = \mathrm{d}b_0/\mathrm{d}y$ is the normalized Brunt–Väisälä frequency squared; and $c = \omega/k$ is the phase speed of the wave.

A solution of (2.3) can be obtained by means of the Frobenius method as a series in the vicinity of the critical point y_c, where $U_0(y_c) = c$. If the Richardson number at the critical point $Ri = N^2(y_c)/(U_y(y_c))^2 > \frac{1}{4}$, it follows that

$$\Psi_1 = Af(y)\,(y-y_c)^{\frac{1}{2}+\mathrm{i}\mu} + Bg(y)\,(y-y_c)^{\frac{1}{2}-\mathrm{i}\mu}, \qquad (2.5)$$

with $f(y_c) = g(y_c) = 1$.

According to (2.4), b_1 is defined as

$$b_1 = -N^2(y_c)/U_0'(y_c)\,(Af_1(y)\,(y-y_c)^{-\frac{1}{2}+\mathrm{i}\mu} + Bg_1(y)\,(y-y_c)^{-\frac{1}{2}-\mathrm{i}\mu}), \qquad (2.6)$$

with $f_1(y_c) = g_1(y_c) = 1$.

Equation (2.4) has singularity points y_c giving rise to branch points in its solutions (2.5), (2.6). This means that (2.3) is invalid in the vicinity of the CL. Some extra factors, therefore, have to be taken into account here. A lot of papers are known to describe separately viscosity and thermal conductivity, nonlinear or non-stationarity in the CL vicinity (see the references in the Introduction). In the present paper the combined effect of dissipation and nonlinearity on a stationary wave is considered.

Since the Reynolds number of the background flow (Re_0) is large and the wave amplitude ϵ is small, it is possible to use the method of matched asymptotic expansions taking into consideration dissipation and nonlinearity only very close to the CL (its scale is defined below) to obtain the inner solution. The wave-field asymptotic form far from the CL is determined by the solutions of the linear inviscid equations (2.5), (2.6), i.e. the outer solution. And the inner solution is used to yield the rules for removing the singularities from the outer solution.

3. The nonlinear viscous-diffusion critical layer

An asymptotic expression for the outer solution at small $|y-y_c|$ can be easily obtained if solution (2.2) is presented as series in $z = y-y_c$:

$$\Psi_\pm = \epsilon^{\frac{2}{3}}c_\pm z+\tfrac{1}{2}u_\pm z^2+\dots + \epsilon\sum_{n=1}^{\infty}\left(Re\left[\left[C^n_\pm|z|^{\frac{1}{2}+i\mu_\pm}(1+\alpha_1^n z+\dots)+D^n_\pm|z|^{\frac{1}{2}-i\mu_\pm}(1+\alpha_2^n z+\dots)\right]e^{i\xi n}\right]\right),$$

$$b_\pm = \epsilon^{\frac{2}{3}}\beta_\pm - N^2_\pm z+\dots \mp \epsilon\frac{N^2_\pm}{u_\pm} \times \sum_{n=1}^{\infty}\left(Re\left[\left[C^n_\pm|z|^{-\frac{1}{2}+i\mu_\pm}(1+\beta_1^n z+\dots)+D^n_\pm|z|^{-\frac{1}{2}-i\mu_\pm}(1+\beta_2^n z+\dots)\right]e^{i\xi n}\right]\right). \qquad (3.1)$$

Signs $(+)$ and $(-)$ respectively refer to the solutions above and below the CL, $\xi = k(x-ct)$ is a normalized horizontal intrinsic coordinate.

The terms $\frac{1}{2}u_\pm z^2$ represent a vorticity jump across the CL which arises similarly to that obtained by Haberman (1972, 1973). But unlike those papers the vorticity jump here arises at the zeroth order of ϵ. Also the jumps in the average velocity and density are taken into account in (3.1). They are expressed by the terms $\epsilon^{\frac{2}{3}}c_\pm z$ in $\Psi_\pm$ and the terms $\epsilon^{\frac{2}{3}}\beta_\pm$ in $b_\pm$ respectively. Similar terms were introduced by Haberman (1973) for a slightly stratified flow. And in the present paper, as in Haberman's, the jumps in velocity and density are of a higher order in ϵ than the vorticity jump. The task consists in determining the connection between the values with subscripts $(+)$ and $(-)$, i.e. the rules for crossing the CL, and yielding the high-harmonic amplitudes.† For this purpose the inner solution has to be found. An obvious result from (3.1) is that the vertical inner coordinate should be defined as follows: $\eta = z/\epsilon^{\frac{2}{3}}$ (see for example Maslowe 1972; Tung, Ko & Chang 1981). Then (3.1) immediately takes the form

$$\left.\begin{aligned} \Psi &= \epsilon^{\frac{4}{3}}\varphi_0+\epsilon^2\varphi_1+\dots, \\ b &= \epsilon^{\frac{2}{3}}b_0+\epsilon^{\frac{4}{3}}b_1+\dots . \end{aligned}\right\} \qquad (3.2)$$

The asymptotic values φ_0 and b_0 for $\eta\to\pm\infty$ ($\varphi_{0\pm}, b_{0\pm}$) are obtained from (3.1). They have the following form:

$$\varphi_{0\pm} = \eta c_\pm+\tfrac{1}{2}u_\pm\eta^2+\sum_{n=1}^{\infty}\mathrm{Re}\left(\left[A^n_\pm|\eta|^{\frac{1}{2}+i\mu_\pm}+B^n_\pm|\eta|^{\frac{1}{2}-i\mu_\pm}\right]e^{i\xi n}\right), \qquad (3.3a)$$

$$b_{0\pm} = \beta_\pm - N^2_\pm\eta \mp \frac{N^2_\pm}{u_\pm}\sum_{n=1}^{\infty}\mathrm{Re}\left(\left[A^n_\pm|\eta|^{-\frac{1}{2}+i\mu_\pm}+B^n_\pm|\eta|^{-\frac{1}{2}-i\mu_\pm}\right]e^{i\xi n}\right). \qquad (3.3b)$$

† Note, that in (3.1) the Brunt–Väisälä frequencies are formally set unequal above and below the CL ($N^2_\pm$); however, it is shown below that N^2_+ and N^2_- appear to be equal.

Here $A^n_\pm = C^n_\pm \epsilon^{\frac{2}{3}i\mu_\pm}$; $B^n_\pm = B^n_\pm \epsilon^{-\frac{2}{3}i\mu_\pm}$; $m_\pm = (N^2_\pm/u^2_\pm - 0.25)^{\frac{1}{2}}$. One more normalization is introduced to simplify the following calculations, namely: $\varphi = (\varphi_0 - c_+\eta)/u_-$; $b = (b_0 - \beta_+)/N^2_-$.

Substituting (3.2) into the system (2.1) and reserving only terms of lowest order in ϵ yields the system for φ and b:

$$\frac{\partial^3\varphi}{\partial\xi\,\partial\eta^2}\frac{\partial\varphi}{\partial\eta} - \frac{\partial^3\varphi}{\partial\eta^3}\frac{\partial\varphi}{\partial\xi} - Ri\frac{\partial b}{\partial\xi} = \lambda\frac{\partial^4\varphi}{\partial\eta^4}, \tag{3.4a}$$

$$\frac{\partial b}{\partial\xi}\frac{\partial\varphi}{\partial\eta} - \frac{\partial b}{\partial\eta}\frac{\partial\varphi}{\partial\xi} = \frac{\lambda}{Pr}\frac{\partial^2 b}{\partial\eta^2}. \tag{3.4b}$$

Here $Ri = N^2_-/u^2_-$ is the Richardson number for $\eta \to -\infty$, and $\lambda = (Re_0\, u_-\, \epsilon^2)^{-1}$ is the parameter characterizing the ratio of viscosity to nonlinearity. To be more exact, $\lambda = (\delta_{\text{vis}}/\delta_{\text{nl}})^3$, where $\delta_{\text{vis}} = (Re_0\, u_-)^{-\frac{1}{3}}$ is the scale of a 'viscous' CL (see for example Hazel 1967), and $\delta_{\text{nl}} = \epsilon^{\frac{2}{3}}$ is the scale of a nonlinear CL (Maslowe 1972; Tung *et al.* 1981). It follows from (3.4) that λ has the sense of a vertical inverse Reynolds number ($Re_i = \lambda^{-1}$) determined by the amplitude of the wave disturbance in the vicinity of the CL. The limit $\lambda \to \infty$ ($Re \ll 1$) corresponds to a viscous linear flow in the vicinity of the CL, and $\lambda \to 0$ ($Re_i \gg 1$) corresponds to a nonlinear non-viscous one.

In general, the system (3.4) should be solved to determine the rules for crossing the CL, but some relations may be obtained without finding the solution.

Thus an expression for the jump in vorticity can be obtained merely by considering the wave and viscous momentum fluxes. Equation (3.4*a*) may be integrated with respect to η from $-\infty$ to ∞ and with respect to ξ from 0 to 2π. Taking into account the ξ-periodicity of the solution gives

$$\int_0^{2\pi} \frac{\partial^2\varphi}{\partial\xi\,\partial\eta}\frac{\partial\varphi}{\partial\eta} - \frac{\partial^2\varphi}{\partial\eta^2}\frac{\partial\varphi}{\partial\xi}\,d\xi = \lambda\frac{d^3}{d\eta^3}\int_0^{2\pi}\varphi\,d\xi. \tag{3.5}$$

Then, integrating (3.5) with respect to η from $-\infty$ to $+\infty$ yields

$$-\int_0^{2\pi}\frac{\partial\varphi}{\partial\eta}\frac{\partial\varphi}{\partial\xi}\,d\xi\,\Big|_{-\infty}^{\infty} = \lambda\frac{d}{d\eta}\int_0^{2\pi}\varphi\,d\xi\,\Big|_{-\infty}^{\infty}. \tag{3.6}$$

Equation (3.6) shows that the radiation force (which is equal to the difference in the wave momentum fluxes above and below the CL (left-hand side of (3.6))) is compensated in the stationary flow by the viscous force which is equal to the difference in viscous stresses (the right-hand side of (3.6)). It results in the formation of a flow in which vorticities tend to some generally different constants ($u_\pm$) far from the CL. The difference in the vorticities, determined by the wave momentum fluxes, obviously depends on the amplitudes of the wave harmonics above and below the CL (which are determined by the boundary conditions at infinity), i.e. it finally depends on the way the problem is set.

Substituting the asymptotic expression (3.3*a*) into (3.6) and making a simple transformation gives

$$2\lambda(u_+ - u_-) + \mu_+\sum_{n=1}^{\infty}(|A^n_+|^2 - |B^n_+|^2) - \mu_-\sum_{n=1}^{\infty}(|A^n_-|^2 - |B^n_-|^2) = 0, \tag{3.7}$$

where $u_- = 1$.

Further, we consider the problem of reflection and transmission of a unit-amplitude fundamental wave disturbance propagating toward the CL from the region $\eta > 0$. The following amplitudes of the wave asymptotic forms for $\eta \to \pm\infty$ correspond to the case

$$|B_+^1| = 1; \quad A_+^1 = R; \quad B_-^1 = T; \quad A_-^1 = 0; \quad A_+^n = R_n; \quad B_-^n = T_n; \quad B_+^n = A_-^n = 0 \quad \text{for} \quad n = 2, \ldots, \tag{3.8}$$

where R, T are reflection and transmission coefficients of the fundamental mode. R_n, T_n are the nth harmonics of the reflected and transmitted fields, which are the amplitudes of the high harmonics radiated by the CL which are normalized by the amplitude of the incident wave of the fundamental mode. The equality of B_+^n and A_-^n to zero means that there are no high harmonics propagating toward the CL.

Under these conditions (3.7) takes the form

$$2\lambda(u_+ - 1) - \mu_+(1 - |R|^2) + \mu_-|T|^2 - \mu_+ \sum_{n=2}^{\infty} |R_n|^2 + \mu_- \sum_{n=2}^{\infty} |T_n|^2 = 0. \tag{3.9}$$

Consider now the consequence of (3.4*b*) for density. Integrating (3.4*b*) with respect to η from $-\infty$ to ∞ and with respect to ξ from 0 to 2π and taking into account the ξ-periodicity of the solution gives

$$\int_0^{2\pi} b \frac{\partial \varphi}{\partial \xi} \mathrm{d}\xi \bigg|_{-\infty}^{\infty} = \frac{\lambda}{Pr} \frac{\mathrm{d}}{\mathrm{d}\eta} \int_0^{2\pi} b \, \mathrm{d}\xi \bigg|_{-\infty}^{\infty}. \tag{3.10}$$

Equation (3.10) together with (3.8) gives $N_+^2 = N_-^2$, where $N_-^2 = 1$, according to normalization, because of the equality of the wave mass fluxes above and below the CL, which are equal to zero. So the diffusion mass fluxes and consequently the density gradients are also equal.

Thus, the values of Brunt–Väisälä frequencies above and below the CL are equal, but the values of the velocity shear are not. This means that the Richardson numbers are different. Under accepted normalizing conditions $Ri_+ = Ri/u_+^2$ above the CL and $Ri_- = Ri$ below the CL.

Simple expressions for the jumps across the CL in the average velocity and density that include only amplitudes of the wave asymptotics cannot be obtained. However, double integrating (3.5) with respect to η and taking into account (3.9) yields

$$c_+ - c_- = -\frac{1}{2\lambda} \int_{-\infty}^{\infty} \eta \frac{\mathrm{d}F}{\mathrm{d}\eta} \mathrm{d}\eta, \tag{3.11}$$

where

$$F = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \varphi}{\partial \eta} \frac{\partial \varphi}{\partial \xi} \mathrm{d}\xi$$

is the wave momentum flux.

An expression for the density jump can be obtained in a similar manner to (3.11):

$$\beta_+ - \beta_- = -\frac{1}{2\lambda} \int_{-\infty}^{\infty} \eta \frac{\mathrm{d}B}{\mathrm{d}\eta} \mathrm{d}\eta. \tag{3.12}$$

Here $B = (1/2\pi) \int_0^{2\pi} b(\partial\varphi/\partial\xi) \, \mathrm{d}\xi$ is the wave mass flux; it differs from zero in the CL vicinity.

4. The spectral model of the nonlinear viscous-diffusion critical layer

Expression (3.9) determines the relation between the amplitudes of the asymptotic values of the wave field and parameters of the mean flow above and below the CL. To calculate the terms of (3.9) and also the velocity and density jumps, the solution of system (3.4) should be obtained. The analytic treating of (3.4) seemed to be unmanageable, so it was evaluated numerically.

Representing (3.4a) in the form of two equations yields a nonlinear system of three equations of the second order for the stream function φ, density b and vorticity χ:

$$\left.\begin{aligned} \frac{\partial\chi}{\partial\xi}\frac{\partial\varphi}{\partial\eta}-\frac{\partial\chi}{\partial\eta}\frac{\partial\varphi}{\partial\xi}-Ri\frac{\partial b}{\partial\xi} &= \lambda\frac{\partial^2\chi}{\partial\eta^2}, \\ \frac{\partial b}{\partial\xi}\frac{\partial\varphi}{\partial\eta}-\frac{\partial b}{\partial\eta}\frac{\partial\varphi}{\partial\xi} &= \frac{\lambda}{Pr}\frac{\partial^2 b}{\partial\eta^2}, \\ \chi &= \frac{\partial^2\varphi}{\partial\eta^2}. \end{aligned}\right\} \tag{4.1}$$

Since the solutions of (4.1) are ξ-periodical, functions χ, b, φ may be represented as Fourier series expansions:

$$\chi(\xi,\eta), b(\xi,\eta), \varphi(\xi,\eta) = \chi_0(\eta), b_0(\eta), \varphi_0(\eta)+\frac{1}{2}\sum_{j=1}^{M}\chi_j(\eta), b_j(\eta), \varphi_j(\eta)\,\mathrm{e}^{\mathrm{i}j\xi}. \tag{4.2}$$

Here $\chi_{-j}, b_{-j}, \varphi_{-j} = \chi_j^*, b_j^*, \varphi_j^*$, (* means complex conjugation), since the functions χ, b, φ are real. There is an infinite number M of series terms in the expansions (4.2) of the exact solutions of (4.1), but since the amplitudes of the harmonics diminish with growth of their number j, taking into consideration a finite number of harmonics will suffice. (The relation of the amplitudes of high harmonics to the fundamental one is discussed below.) This makes it possible to realize a numerical spectral model consisting of a finite number of harmonics. Substitution of (4.2) into (4.1) gives a system of three equations for the average components χ_0, b_0, φ_0, and $3M$ nonlinear ordinary differential equations for complex amplitudes of harmonics χ_j, b_j, φ_j:

$$\lambda\frac{\mathrm{d}^2\chi_0}{\mathrm{d}\eta^2} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\sum_{j=1}^{M}\mathrm{Im}\,(j\chi_j\,\varphi_j^*), \tag{4.3a}$$

$$\frac{\lambda}{Pr}\frac{\mathrm{d}^2 b_0}{\mathrm{d}\eta^2} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\sum_{j=1}^{M}\mathrm{Im}\,(jb_j\,\varphi_j^*), \tag{4.3b}$$

$$\frac{\mathrm{d}^2\varphi_0}{\mathrm{d}\eta^2} = \chi_0; \tag{4.3c}$$

$$\frac{\lambda}{Pr}\frac{\mathrm{d}^2 b_j}{\mathrm{d}\eta^2} = \mathrm{i}j(\varphi_{0\eta}b_j - b_{0\eta}\varphi_j)+S_j(b,\varphi), \tag{4.4a}$$

$$\lambda\frac{\mathrm{d}^2\chi_j}{\mathrm{d}\eta^2} = \mathrm{i}j(\varphi_{0\eta}\chi_j - \chi_{0\eta}\varphi_j - Rib_j)+S_j(\chi,\varphi) \tag{4.4b}$$

$$\frac{\mathrm{d}^2\varphi_j}{\mathrm{d}\eta^2} = \chi_j, \quad j = 1,\ldots,M. \tag{4.4c}$$

Here
$$S_j(a,\varphi) = \tfrac{1}{2}\mathrm{i}\sum_{m=1}^{j-1}(j-m)\left(\alpha_{(j-m)}\frac{\mathrm{d}\varphi_m}{\mathrm{d}\eta}-\varphi_{(j-m)}\frac{\mathrm{d}\alpha_m}{\mathrm{d}\eta}\right)$$
$$+\tfrac{1}{2}\mathrm{i}\sum_{m=1}^{M-j}(j+m)\left(\alpha_{(j+m)}\frac{\mathrm{d}\varphi^*_m}{\mathrm{d}\eta}-\varphi_{(j+m)}\frac{\mathrm{d}\alpha^*_m}{\mathrm{d}\eta}\right)$$
$$-\tfrac{1}{2}\mathrm{i}\sum_{m=1}^{M-j}m\left(\alpha^*_m\frac{\mathrm{d}\varphi_{(j+m)}}{\mathrm{d}\eta}-\varphi^*_m\frac{\mathrm{d}\alpha_{(j+m)}}{\mathrm{d}\eta}\right). \qquad (4.4d)$$

The components $S_j(a,\varphi)$ are determined by nonlinear terms. Taking into account the asymptotic behaviour of the wave fields at infinity, integration of (4.3*a*, *b*) yields

$$\chi_{0\eta} = -\frac{1}{2\lambda}\mathrm{Im}\sum_{j=1}^{M}\mathrm{Im}\,(j\chi_j\varphi^*_j), \qquad (4.5a)$$

$$b_{0\eta} = -1-\frac{Pr}{2\lambda}\mathrm{Im}\sum_{j=1}^{M}\mathrm{Im}\,(jb_j\varphi^*_j). \qquad (4.5b)$$

Taking into account that $\chi_0(-\infty) = 1$, integration of (4.5*a*) gives

$$\chi_0 = 1-\frac{1}{2\lambda}\mathrm{Im}\left[\sum_{j=1}^{M}\left(j\frac{\mathrm{d}\varphi_j}{\mathrm{d}\eta}\varphi^*_j\right)-\sum_{j=1}^{M}\left(j\frac{\mathrm{d}\varphi_j}{\mathrm{d}\eta}\varphi^*_j\right)\Bigg|_{\eta=-\infty}\right] \qquad (4.5c)$$

The boundary problem for the system (4.4) has been solved to find the amplitudes of the harmonics χ_j, b_j, φ_j. Matching with the asymptotic form (3.3) of the wave fields completed by the expression for the vorticity field ($\chi = \varphi_{\eta\eta}$) has been used as the boundary conditions. In practice these boundary conditions have been realized in the following way. For $\eta = z_0 > 0$ (where $|z_0| \gg \max\{\delta_{\mathrm{vis}}, 1\}$) the complex amplitudes of the fundamental harmonics of the wave fields have been determined. Namely,

$$\varphi_1 = z_0^{\frac{1}{2}}, \quad b_1 = \frac{\varphi_1 b_{0\eta}(\infty)}{\chi_0(\infty)z_0}, \quad \chi_1 = -\frac{Ri\varphi_1}{\chi_0^2(\infty)z_0^2}. \qquad (4.6)$$

For the higher harmonics the radiation conditions were formulated:

$$\left.\begin{aligned}\frac{\mathrm{d}\varphi_j}{\mathrm{d}\eta}\bigg|_{\eta=z_0} &= (\tfrac{1}{2}+\mathrm{i}\mu_+)\frac{\varphi_j(z_0)}{z_0},\\ \frac{\mathrm{d}b_j}{\mathrm{d}\eta}\bigg|_{\eta=z_0} &= -(\tfrac{1}{2}-\mathrm{i}\mu_+)\frac{b_j(z_0)}{z_0},\\ \frac{\mathrm{d}\chi_j}{\mathrm{d}\eta}\bigg|_{\eta=z_0} &= -(\tfrac{3}{2}-\mathrm{i}\mu_+)\frac{\chi_j(z_0)}{z_0}.\end{aligned}\right\} \qquad (4.7a)$$

This means that there are no high wave harmonics propagating towards the CL and the only ones radiated by the CL occur when $\eta = z_0$. It should be mentioned that the results of the numerical calculations show that the high-harmonic amplitudes are extremely small for all the parameters of the problem under consideration. For example the maximal second harmonic amplitude is of order 10^{-3} with respect to the fundamental one. Taking this into account enables the radiation conditions (4.7*a*) to be replaced by the zero boundary conditions for $\eta = z_0$, giving better stability of the numerical model. Namely

$$\varphi_j(z_0) = b_j(z_0) = \chi_j(z_0) = 0 \quad \text{for} \quad j = 2, \ldots M. \tag{4.7b}$$

The conditions (4.7*b*) are clearly equivalent to the assumption that the extremely weak high-harmonic waves propagates toward the CL provide the zero boundary conditions at $\eta = z_0$. As numerical estimates show, the differences between the CL characteristics (reflection and transmission coefficients, jumps in velocity, density and vorticity) yielded under (4.7*a*) and (4.7*b*) boundary conditions do not exceed 1%.

It should be mentioned that the amplitude of the incident wave differs from unity when the amplitudes of the first harmonics are defined by (4.6), i.e. in this case $|B_+| \neq 1$; $A_+ = RB_+$; $B_- = TB_+$. The inner variables should be renormalized to give an incident wave of unit amplitude:

$$\eta = \frac{\eta_{\text{old}}}{|B_+|^{\frac{2}{3}}}; \quad \varphi = \frac{\varphi_{\text{old}}}{|B_+|^{\frac{4}{3}}}; \quad b = \frac{b_{\text{old}}}{|B_+|^{\frac{2}{3}}}.$$

Parameter λ is also renormalized to

$$\lambda = \lambda_{\text{old}}/|B_+|^2. \tag{4.8}$$

It should be emphasized that λ but not λ_{old} is the term in (3.9). The jumps in velocity and density are renormalized in the following way:

$$(c_+ - c_-) = \frac{(c_+ - c_-)_{\text{old}}}{|B_+|^{\frac{2}{3}}}, \quad (\beta_+ - \beta_-) = \frac{(\beta_+ - \beta_-)_{\text{old}}}{|B_+|^{\frac{2}{3}}},$$

and the phase of the reflection coefficient changes, namely

$$\phi_R(\lambda) = \arg(R) = \arg(R_{\text{old}}) + \tfrac{4}{3}\mu_+ \ln|B_+|.$$

When $\eta = z_1$, where $z_1 < 0$ (just as $|z_0|, |z_1| > \max\{\delta_{\text{vis}}, 1\}$), the wave fields satisfy the radiation conditions, the form of which follows from the asymptotic expressions (3.3):

$$\left.\begin{aligned} \left.\frac{d\varphi_1}{d\eta}\right|_{\eta=z_1} &= (\tfrac{1}{2} - i\mu_-)\frac{\varphi_1(z_1)}{z_1}, \\ \left.\frac{db_1}{d\eta}\right|_{\eta=z_1} &= -(\tfrac{1}{2} + i\mu_-)\frac{b_1(z_1)}{z_1}, \\ \left.\frac{d\chi_1}{d\eta}\right|_{\eta=z_1} &= -(\tfrac{3}{2} - i\mu_-)\frac{\chi_1(z_1)}{z_1}. \end{aligned}\right\} \tag{4.9}$$

The zero boundary conditions for $\eta = z_1$, as for $\eta = z_0$, can be set for the high harmonics since their amplitudes are extremely small.

The problem has been solved by means of the grid method. The step size (h) of the grid has been chosen to be much less than δ_{vis} and 1. The calculations were carried out for $h = 0.02$ and $h = 0.04$. The differences of the calculated values in this two cases were of order 10^{-3}. The finite-difference approximation of the system of differential equations (4.3) leads to a system of algebraic equations which may be written in matrix form:

$$\hat{\boldsymbol{A}}_j(Y)\,Y_j = \boldsymbol{B}_j(Y). \tag{4.10}$$

Here Y_j is a column vector of the functions χ_j, b_j, φ_j at N points (nodes of the grid). Namely:

$$b_j(\eta_i) = b_j^{(i)} = Y_j(3i-2), \quad \chi_j(\eta_i) = Y_j(3i-1), \quad \varphi_j(\eta_i) = \varphi_j^{(i)} = Y_j(3i).$$

$\hat{\boldsymbol{A}}_j$ is a band matrix $(3M\times 3M)$ with band width equal to seven. When the uniform grid is used the elements of matrix $\hat{\boldsymbol{A}}_j$ according to (4.3), (4.7b) for $i = 1,\dots,N$, $j = 2,\dots,M$ and $i = 1,\dots,N-1$, $j = 1$ are

$A(3i-2,1) = \lambda/Pr,$	$A(3i-1,1) = \lambda,$	$A(3i,1) = 1,$
$A(3i-2,2) = 0,$	$A(3i-1,2) = 0,$	$(3i,2) = 0,$
$A(3i-2,3) = 0,$	$A(3i-1,3) = ijh^2Ri,$	$A(3i,3) = -h^2,$
$A(3i-2,4) = -2\lambda/Pr-ijh^2\varphi_{0\eta}^{(i)},$	$A(3i-1,4) = -2\lambda-ijh^2\varphi_{0\eta}^{(i)},$	$A(3i,4) = -2,$
$A(3i-2,5) = 0,$	$A(3i-1,5) = ijh^2\chi_{0\eta}^{(i)},$	$A(3i,5) = 0,$
$A(3i-2,6) = ijh^2b_{0\eta}^{(i)},$	$A(3i-1,6) = 0,$	$A(3i,6) = 0,$
$A(3i-2,7) = 0,$	$A(3i-1,7) = \lambda,$	$A(3i,7) = 1.$

The expressions for $\hat{\boldsymbol{A}}_1$ at $i = N$ follow from the finite-difference approximation of the boundary radiation conditions (4.9). They have the form:

$$\begin{aligned}
A(3N-2,1) &= 2\lambda/Pr,\\
A(3N-2,4) &= -2\lambda/Pr - \mathrm{i}h^2\varphi_{0\eta}^{(N)} + 2h(-0.5-\mathrm{i}\mu_-)\,\lambda/Pr/z_1,\\
A(3N-1,1) &= 2\lambda,\\
A(3N-1,4) &= -2\lambda - \mathrm{i}h^2\varphi_{0\eta}^{(N)} + 2h(-1.5-\mathrm{i}\mu_-)\,\lambda/z_1,\\
A(3N,1) &= 2,\\
A(3N,4) &= -2-\mathrm{i}h^2\varphi_{0\eta}^{(N)} + 2h(0.5-\mathrm{i}\mu_-)/z_1.
\end{aligned}$$

The expressions for the elements of the matrix $\hat{\boldsymbol{A}}_j$ corresponding to the boundary conditions (4.7a) can be similarly obtained. The column vector $\boldsymbol{B}_j(Y)$ in (4.10) follows from the finite-difference approximation of the nonlinear terms in (4.4). In this case for $i = 2,\dots,N$

$$\begin{aligned}
B_1(3i-2) &= h^2S_j^b,\\
B_1(3i-1) &= h^2S_j^\chi,\\
B_1(3i) &= 0.
\end{aligned}$$

Here S_j^b and S_j^χ are finite-difference approximations of (4.4d). From the boundary conditions (4.6) the expressions for B_1 follow:

$$\begin{aligned}
B_1(3i-2) &= -\lambda/Prb_1(z_0) + h^2S_j^b,\\
B_1(3i-1) &= -\lambda\chi_1(z_0) + h^2S_j^\chi,\\
B_1(3i) &= -\varphi_1(z_0).
\end{aligned}$$

An iteration method has been employed to solve system (4.8). At every step the algebraic system with the matrix $\hat{\boldsymbol{A}}$ has been solved by means of the Gauss elimination method. The numerical algorithm described by Forsythe & Moler (1967) has been used, modified for the case of a band matrix.

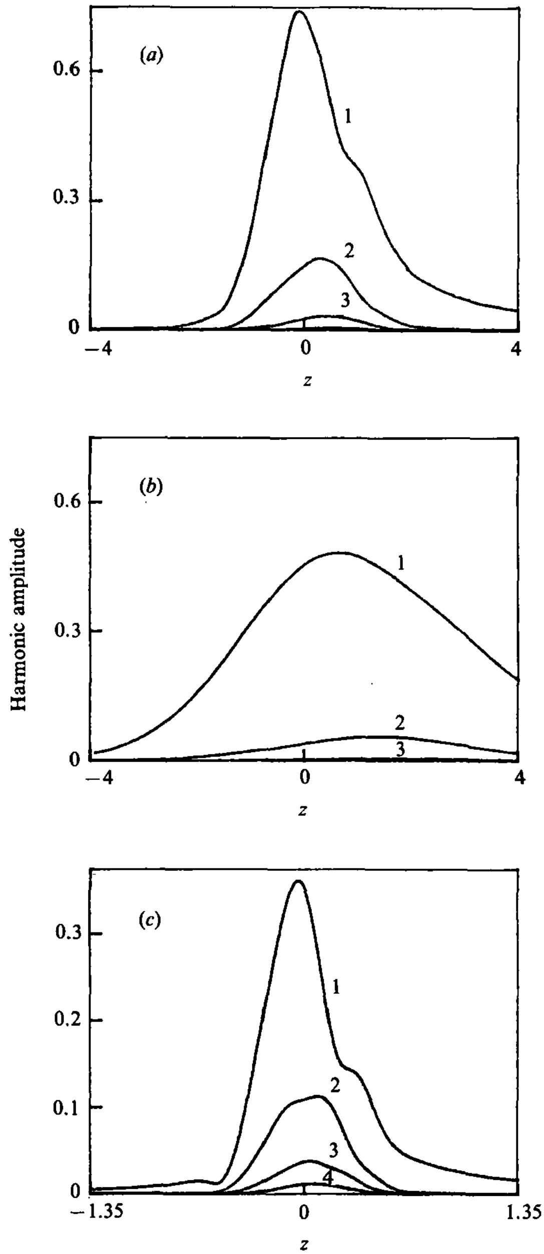

FIGURE 1. The absolute values of the amplitudes of the vorticity-field harmonics. The curves are labelled with the number of the harmonic mode. The parameters of the flow are (*a*) $\lambda = 0.22$, $Ri = 3$; (*b*) $\lambda = 0.91$, $Ri = 3$; (*c*) $\lambda = 0.0016$, $Ri = 1$. $Pr = 0.71$.

Before passing on to a description of the numerical calculation results we shall remark on the applicability of the spectral model for the investigation of the stationary nonlinear viscous-diffusion CL. The spectral model can obviously be successfully applied if usage of a few harmonics suffices to achieve a satisfactory accuracy. As was mentioned above the amplitudes of high-harmonic asymptotics were small with respect to the fundamental one; however, in the CL vicinity the high-harmonic amplitudes are comparable with the fundamental one, but decreasing with their mode number grows. The different characteristics of the CL depend differently on the number of harmonic modes taken into account. Thus only the fundamental mode suffices to calculate the vorticity jump for a wide range of Ri and λ. It follows from (3.9) that if the reflection and transmission coefficients of the fundamental harmonic are small then the vorticity jump can be determined with an accuracy of $|R|^2$, $|T|^2$. However, the structure of the wave field in the CL vicinity (where harmonics are of comparable amplitude) is required to calculate the reflection and transmission coefficients and the velocity and density jumps. So a number of harmonic modes should be taken into consideration. The absolute values of the vorticity-field harmonic amplitudes for some values of parameters Ri and λ are plotted on figure 1. As the mode number of a harmonic grows its amplitude is seen to decrease rapidly, which permits one to use a spectral model consisting of a few harmonics modes for a wide range of parameters Ri and λ. In practice, the number of harmonics modes has been limited if the addition of one more does not change the velocity and density jumps within the 10^{-3} accuracy, and the number of harmonic modes M required did not exceed 10. There is no contradiction here with the review article by Maslowe (1986) since in that work the marginally sufficient number of harmonics $M = 32$ corresponded to an internal Reynolds number $Re_i = 200$.† In our calculations the vertical internal Reynolds number did not exceed 100, so a smaller number of harmonics has appeared to be sufficient. We shall return below to the discussion of the internal Reynolds number for $\lambda \to 0$.

5. The results of numerical calculations

The values of the variables in (3.9), namely u_+, $|R|$, $|T|$, and also the velocity $(c_+ - c_-)$ and density $(\beta_+ - \beta_-)$ jumps, against λ are obtained from numerical calculations: $u_+(\lambda)$, $|R|(\lambda)$, the normalized value $|T| e^{\mu_- \pi}(\lambda)$, the phase of the reflection coefficient $\phi_R(\lambda)$, the values $(c_+ - c_-)(\lambda)$ and $(\beta_+ - \beta_-)(\lambda)$ are plotted respectively on figures 2–7 for four values of the Richardson number, $Ri = 0.5$, 1, 2, 3.

The function $u_+(\lambda)$ may be described by a simple expression for a wide range of λ. Indeed, for all $Ri \geqslant 1$ and $\lambda \geqslant 0.15$, as one can see from figures 3 and 4, the reflection and transition coefficients squared are much less than unity, i.e. $|R|^2 \ll 1$; $|T|^2 \ll 1$. And in this case the following relation results from (3.9):

$$\lambda(u_+) = \frac{(Ri/u_+^2 - 0.25)^{\frac{1}{2}}}{2(u_+ - 1)}. \tag{5.1}$$

On figure 2 the dashed curves indicate the dependence of λ on u_+ calculated according to (5.1). They are seen to coincide with the numerically calculated dependence relationships for almost all λ. Differences exist only in a narrow range near $\lambda = 0$. The small values of λ are typical for waves in the ocean and atmosphere (Tung *et al.* 1981).

It follows both from the numerical calculations and from the expression (5.1) that as λ tends to zero, then the meaning of u_+ tends to the value $2Ri^{\frac{1}{2}}$ for which the Richardson number above the CL (Ri_+) equals $\frac{1}{4}$ and $\mu_+ = 0$. But the character of the relationship $\lambda(u_+)$ which follows from (5.1) differs from that obtained from numerical

† To be more exact it should be mentioned that the calculations reviewed in Maslowe's article were carried out for $Ri_0 = 200$. But one can see from Collins & Maslowe (1988) that in the calculations the CL scale was of the order of unity in the variables of outer flow. In our terms this means that $Ri_i \approx Ri_0$.

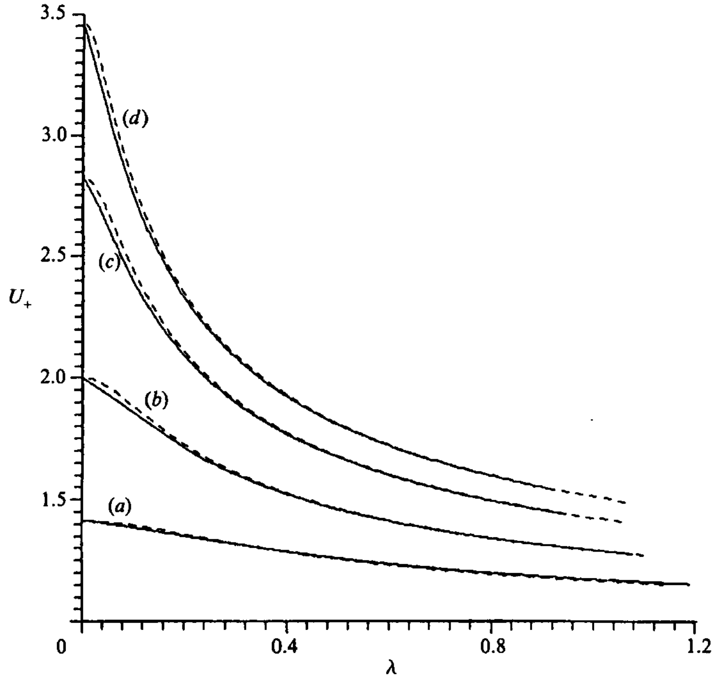

FIGURE 2. The dependence of the vorticity value above the CL, u_+, on λ for the following values of the Richardson number: (*a*) $Ri = 0.5$, (*b*) $Ri = 1$, (*c*) $Ri = 2$, (*d*) $Ri = 3$. $Pr = 0.71$. The dashed curves represent the function $u_+(\lambda)$ calculated by (5.1).

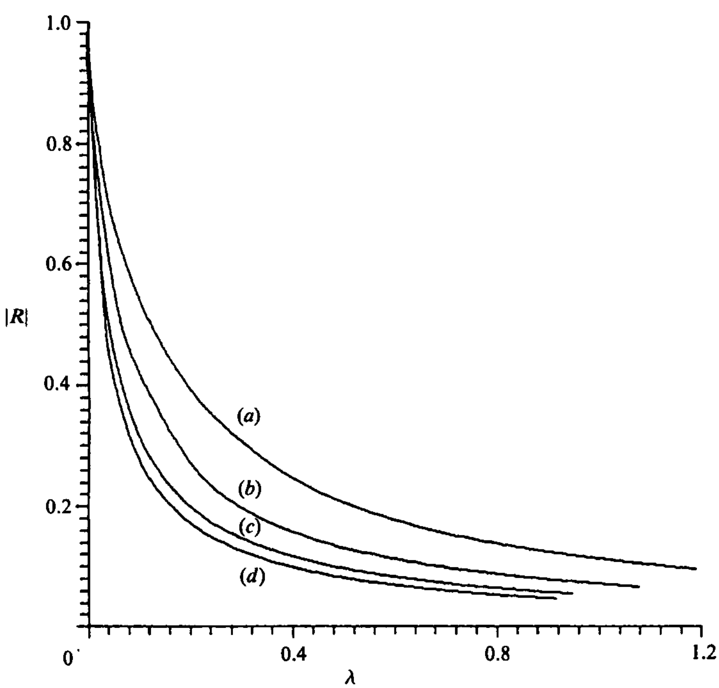

FIGURE 3. The dependence of the absolute value of the reflection coefficient $|R|$ on λ for (*a*) $Ri = 0.5$, (*b*) $Ri = 1$, (*c*) $Ri = 2$, (*d*) $Ri = 3$. $Pr = 0.71$.

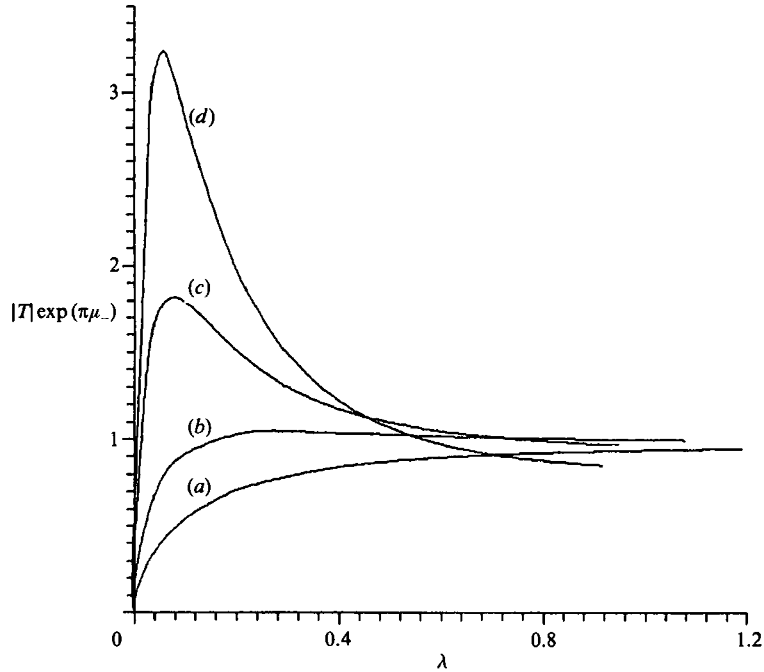

FIGURE 4. The dependence of the normalized value of the transmission coefficient $|T| \exp(\mu_- \pi)$ on λ for (*a*) $Ri = 0.5$, (*b*) $Ri = 1$, (*c*) $Ri = 2$, (*d*) $Ri = 3$. $Pr = 0.71$.

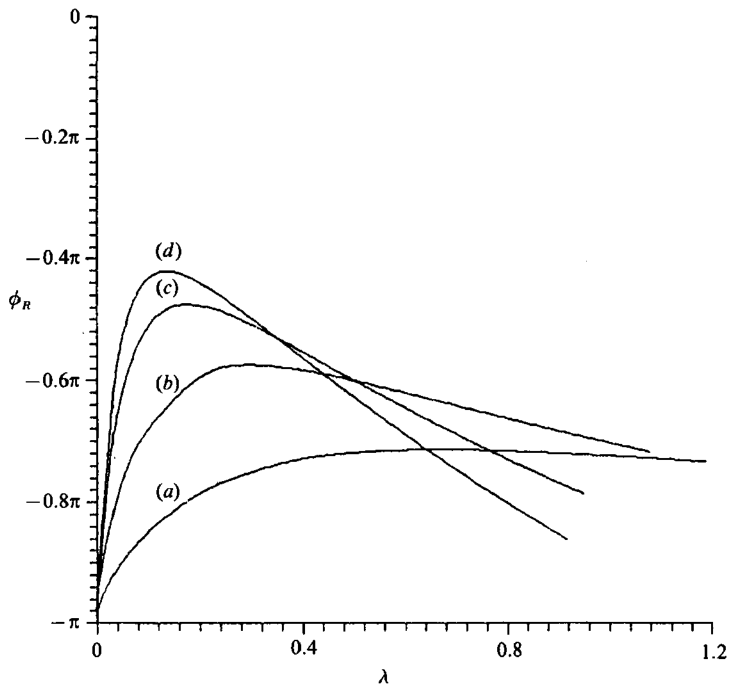

FIGURE 5. The dependence of the phase of the reflection coefficient ϕ_R on λ for (*a*) $Ri = 0.5$, (*b*) $Ri = 1$, (*c*) $Ri = 2$, (*d*) $Ri = 3$. $Pr = 0.71$.

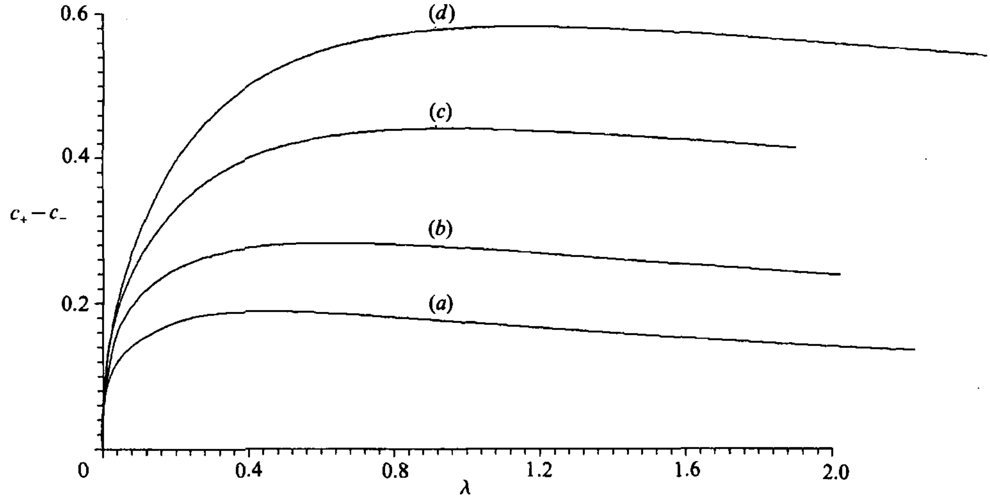

FIGURE 6. The dependence of the velocity jump $c_+ - c_-$ on λ for (*a*) $Ri = 0.5$, (*b*) $Ri = 1$, (*c*) $Ri = 2$, (*d*) $Ri = 3$.

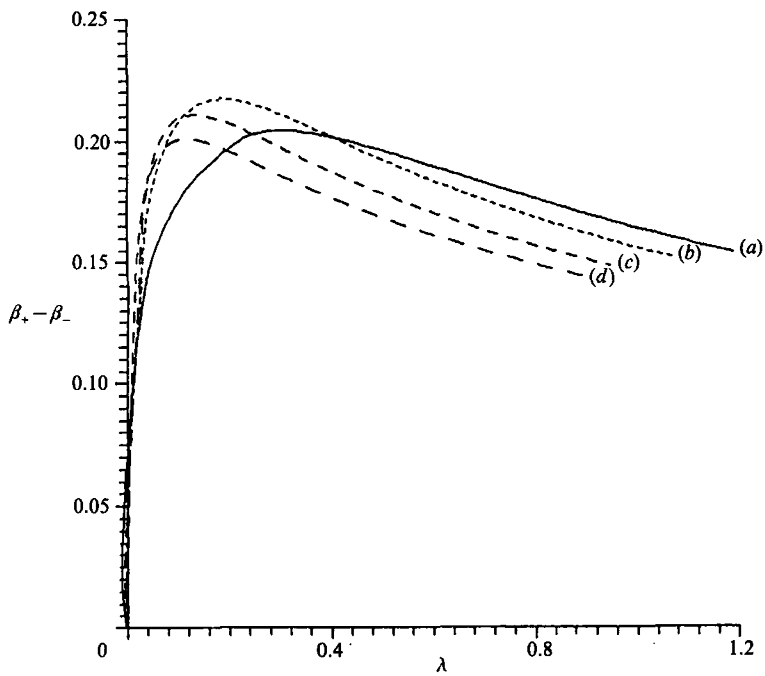

FIGURE 7. The dependence of the density jump $\beta_+ - \beta_-$ on λ for (*a*) $Ri = 0.5$, (*b*) $Ri = 1$, (*c*) $Ri = 2$, (*d*) $Ri = 3$. $Pr = 0.71$.

calculations. When λ tends to zero one can see from figures 3 and 5, and figure 4 respectively that the reflection coefficient R tends to minus unity and the transmission one T tends to zero. This numerical result can be explained by the following qualitative arguments. When Ri_+ tends to $\frac{1}{4}$, the wave field becomes less oscillatory, i.e. the effective vertical wavelength above the CL tends to infinity. At the same time Ri below the CL is not equal to $\frac{1}{4}$ and the wavelength is finite. Thus the situation appears to be similar to the reflection of waves at the boundary of media with strongly different refraction coefficients, where the reflection coefficient

$$R = -1 + O(k_1/k_2),$$

and k_1, k_2 are respectively the wavenumbers of the incident and transmitted waves. In the case under consideration k_1/k_2 is of the order μ_+/μ_- and the reflection coefficient

$$R = -1 + O(\mu_+/\mu_-); \tag{5.2}$$

and the order of the transmission coefficient does not exceed μ_+/μ_-. Taking this into account in (3.9) yields

$$\lambda = O\left(\frac{\mu_+^2}{\mu_-(2Ri^{\frac{1}{2}}-1)}\right). \tag{5.3}$$

For a small μ_+ the complex amplitude of the wave field above the CL is a small value of order μ_+. Indeed

$$\varphi_+ = \eta^{\frac{1}{2}-i\mu_+} + \left(-1 + O\left(\frac{\mu_+}{\mu_-}\right)\right)\eta^{\frac{1}{2}+i\mu_+} \approx \eta^{\frac{1}{2}} O\left(\frac{\mu_+}{\mu_-}\right) - 2i\mu_+\eta^{\frac{1}{2}} \ln \eta.$$

In this case the inner variables determined by the amplitude of the incident wave become unnatural for the CL. The inner variables should be renormalized to turn the natural variables in which the wave fields are of the unity order, namely

$$\eta_{new} = \eta/\mu_+^{\frac{2}{3}}; \quad \varphi_{new} = \varphi/\mu_+^{\frac{4}{3}}; \quad b_{new} = b/\mu_+^{\frac{2}{3}}; \quad \chi_{new} = \chi.$$

Determined in the new variables, the vorticity jump remains the same, but the velocity and density jumps are renormalized in the following way:

$$(c_+ - c_-)_{new} = (c_+ - c_-)/\mu_+^{\frac{2}{3}}, \quad (\beta_+ - \beta_-)_{new} = (\beta_+ - \beta_-)/\mu_+^{\frac{2}{3}}.$$

The fields φ_{new} and b_{new} satisfy system (3.4) in which λ_{new} is of order $\lambda\mu_+^{-2}$. Together with (5.3) this gives

$$\lambda_{new} = O((Ri - 0.25)(2Ri^{\frac{1}{2}} - 1))^{-\frac{1}{2}}. \tag{5.4}$$

It clearly follows from (5.4) that λ_{new} decreases when Ri grows, i.e. the CL becomes 'more nonlinear'. It should be mentioned that the variables with the subscript 'new' are equivalent to those with the subscript 'old' from (4.8) since the amplitudes of the fields in (4.6) expressed in the 'old' variables are of unity order, and λ_{new} is evidently of the same order as λ_{old} from (4.8).

Thus, if the Reynolds number determined by the incident wave amplitude (λ^{-1}) grows, then the value of u_+ tends to $2Ri^{\frac{1}{2}}$ for which $Ri_+ = \frac{1}{4}$. In this case the reflection wave appears to have a reflection coefficient close to unity in the antiphase of the incident wave. This leads to the total wave field (the sum of the incident and reflected wave fields) having an amplitude much less than that of the incident one. Moreover, if the amplitude of the incident wave increases (λ decreases) Ri_+ becomes closer to $\frac{1}{4}$, the reflection coefficient becomes closer to minus unity and the value of the total field remains invariable (λ_{new} does not depend on λ). As a result the flow in the vicinity of the CL does not vary, and the Reynolds number of this flow Re_{new} is determined by the parameter λ_{new} (or by λ_{old} which is just the same) but not by λ. As a result the Reynolds number in the CL vicinity remains fixed and not large when $\lambda \to 0$. Estimated from λ_{old}, for $Ri = 0.5$, $Re_{new} \approx 10$; for $Ri = 1$, $Re_{new} \approx 30$; for $Ri = 2$, $Re_{new} \approx 60$; for $Ri = 3$, $Re_{new} \approx 70$.

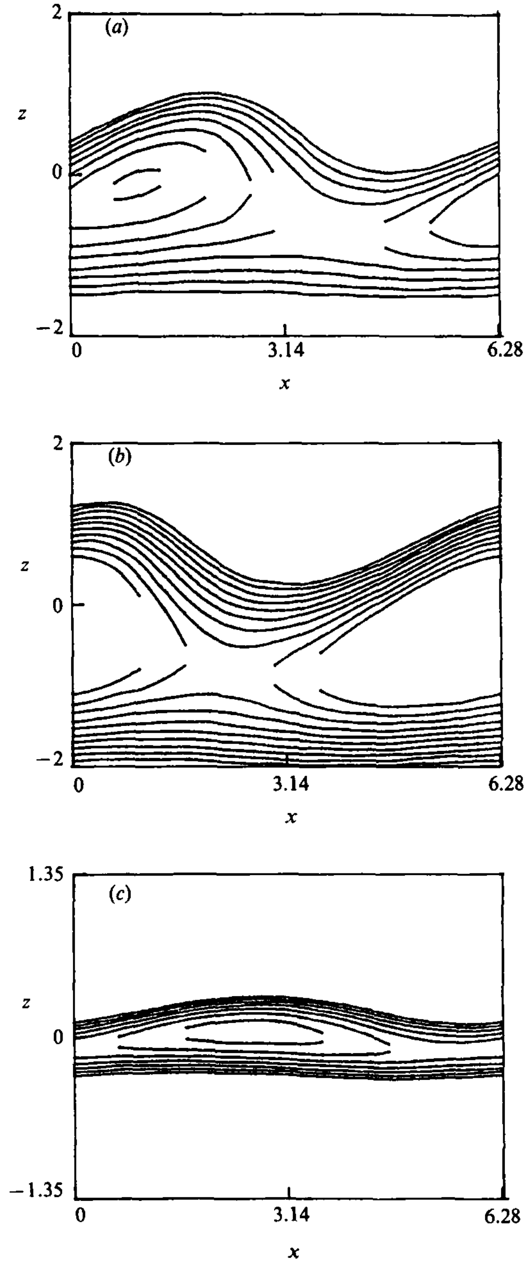

FIGURE 8. The streamline pattern in the vicinity of the CL. The flow parameters are (*a*) $\lambda = 0.22$, $Ri = 3$; (*b*) $\lambda = 0.91$, $Ri = 3$; (*c*) $\lambda = 0.0016$, $Ri = 1$; $Pr = 0.71$.

This result differs from that of Maslowe (1972), where a symmetric pattern of streamlines was constructed for $\lambda = 0$ including the case $Ri_+ > \frac{1}{4}$. This difference is caused by neglecting in that work the jump of vorticity in the mean flow when the solution in the vicinity of the CL is constructed. And it results in the neglect the wave reflected from the CL. At the same time the numerical calculations and qualitative considerations given above show that as λ tends to zero a jump in vorticity occurs, from which the wave reflects with an opposite phase and a reflection coefficient close

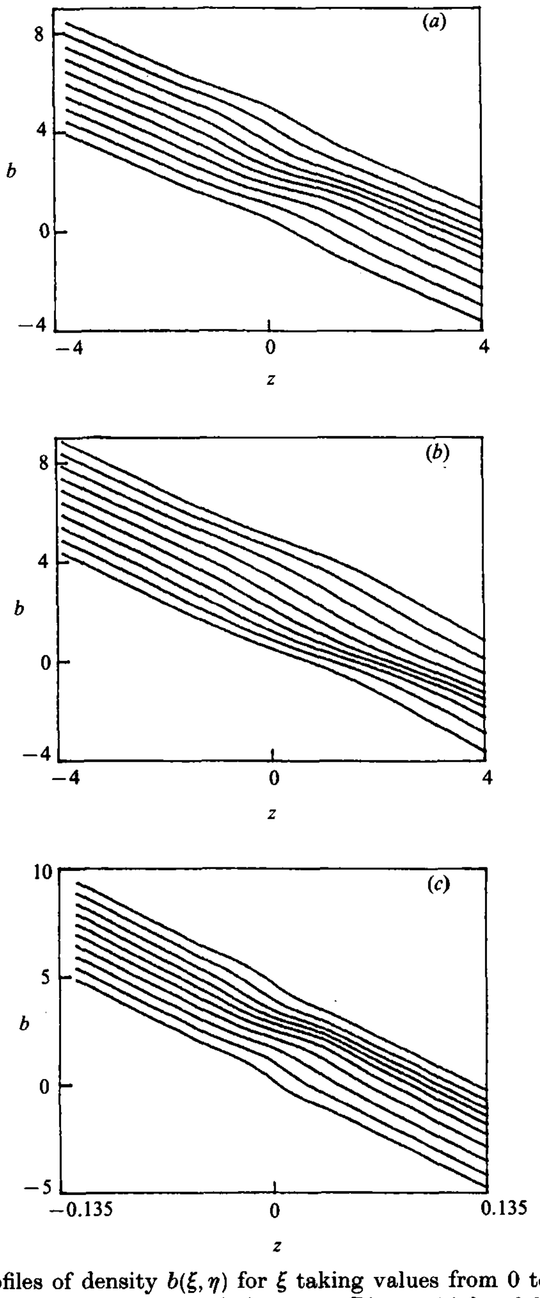

FIGURE 9. Vertical profiles of density $b(\xi, \eta)$ for ξ taking values from 0 to 2π with a step of $\frac{2}{9}\pi$, $Pr = 0.71$. (a) $\lambda = 0.22$, $Ri = 3$; (b) $\lambda = 0.91$, $Ri = 3$; (c) $\lambda = 0.0016$, $Ri = 1$.

to unity. As a result, the wave-field amplitude in the vicinity of the CL becomes much smaller than the amplitude of the incident wave. And the actual 'nonlinearity' becomes considerably smaller than that determined by the parameter λ.

In conclusion we shall discuss some figures illustrating the behaviour of wave fields in the vicinity of the CL. Streamline patterns for some parameters of the flow are shown on figure 8. Sections through the CL of the density field b, the density-gradient field β_η, the velocity field u and the vorticity field χ are plotted on figures 9–12

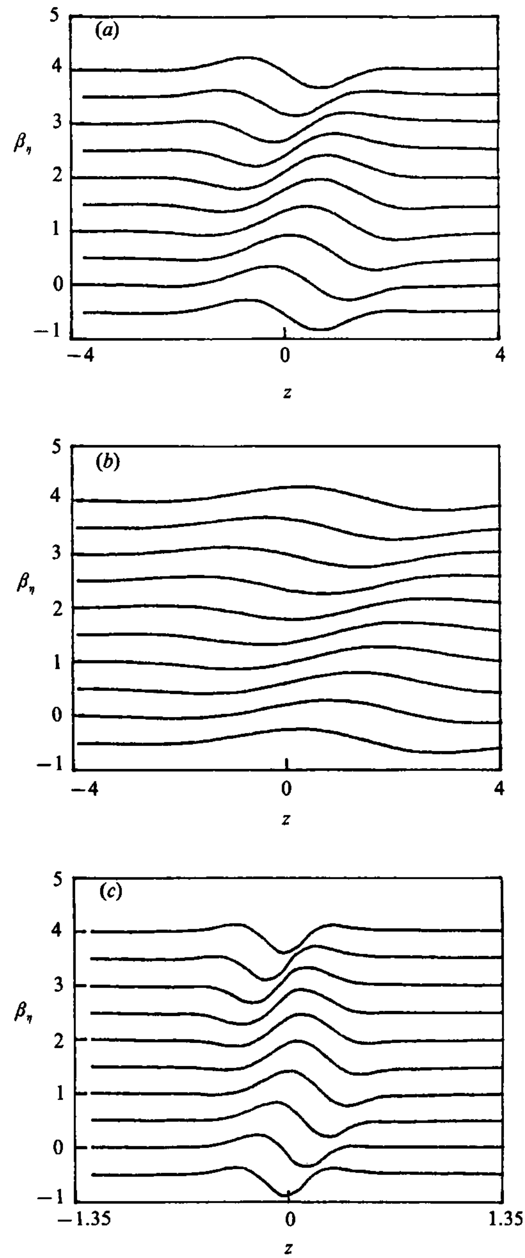

FIGURE 10. Vertical profiles of the density gradient $\beta_\eta(\xi, \eta)$. All the parameters are the same as for figure 9.

respectively, and the average fields $b_{0\eta}$, χ_0, u_0 shown on figure 13. It should be emphasized that inner variables determined by the incident wave amplitude are used everywhere.

The pattern of streamlines in the CL vicinity is a modification of the familiar Kelvin's cat's eye solution; a region of coupled streamlines exists (figure 8). The streamline pattern is non-symmetric about the CL. This is due to the strong attenuation of a wave passing through this layer. The transition from one wave-field

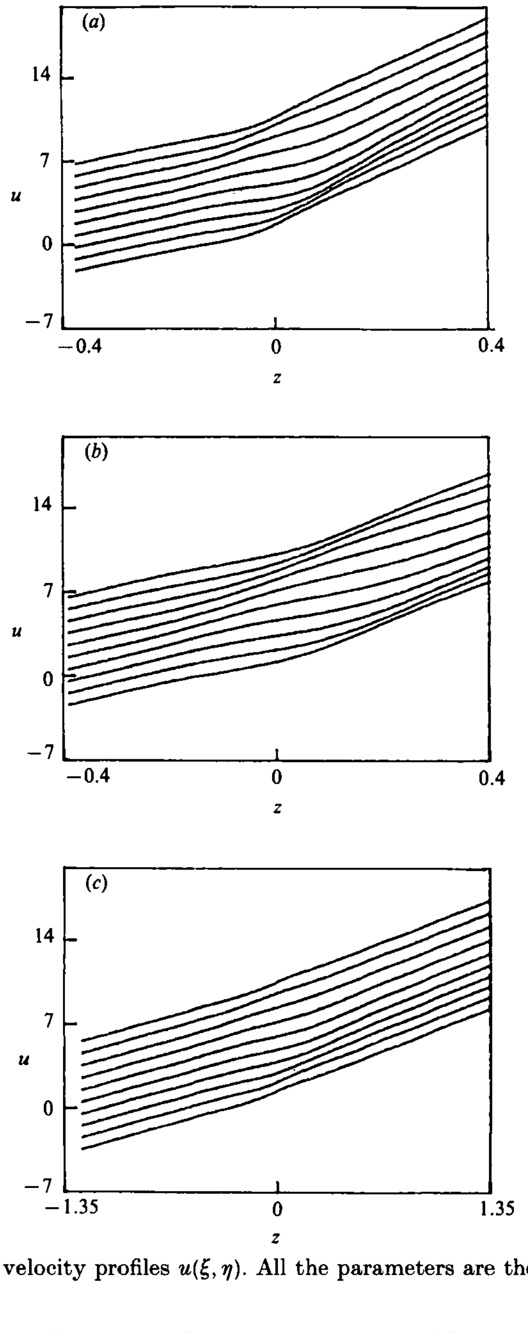

FIGURE 11. Vertical velocity profiles $u(\xi, \eta)$. All the parameters are the same as for figure 9.

asymptotic form to another is seen from a comparison of figure 8 and figures 9–13 and occurs in the region of coupled streamlines.

It should be noted that figure 8(*c*) corresponds to the small value of $\lambda = 0.0016$. In this case the Richardson number above the CL is close to $\frac{1}{4}$. The streamline pattern (figure 8*c*) differs from the symmetric cat's-eye flow: on the one hand, the width of the closed-streamline region is 0.4, on the other hand, since $\lambda = 0.0016$ the viscous scale $\delta_{vis} = \lambda^{\frac{1}{3}} = 0.1$. Thus, in this case viscosity and thermal conductivity are essential factors in the CL.

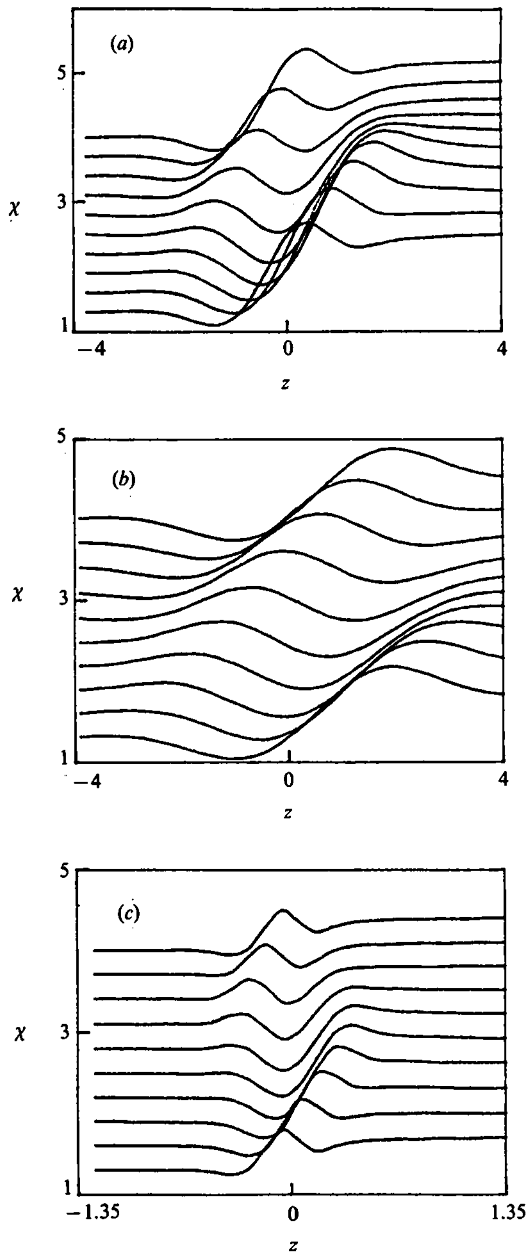

FIGURE 12. Vertical vorticity profiles $\chi(\xi, \eta)$. All the parameters are the same as for figure 9.

6. Conclusions

A numerical investigation of the nonlinear viscous-diffusion stationary CL enables one to understand a number of its qualitative features.

Owing to nonlinear interaction between the wave and the flow the jump in average vorticity across the CL occurs at the zeroth order of the wave amplitude. Its origin is associated with the following. A jump in the vertical flux of the horizontal momentum component of the wave occurs across the CL, which means that a horizontal radiational force acts in the CL and in the stationary case it has to be

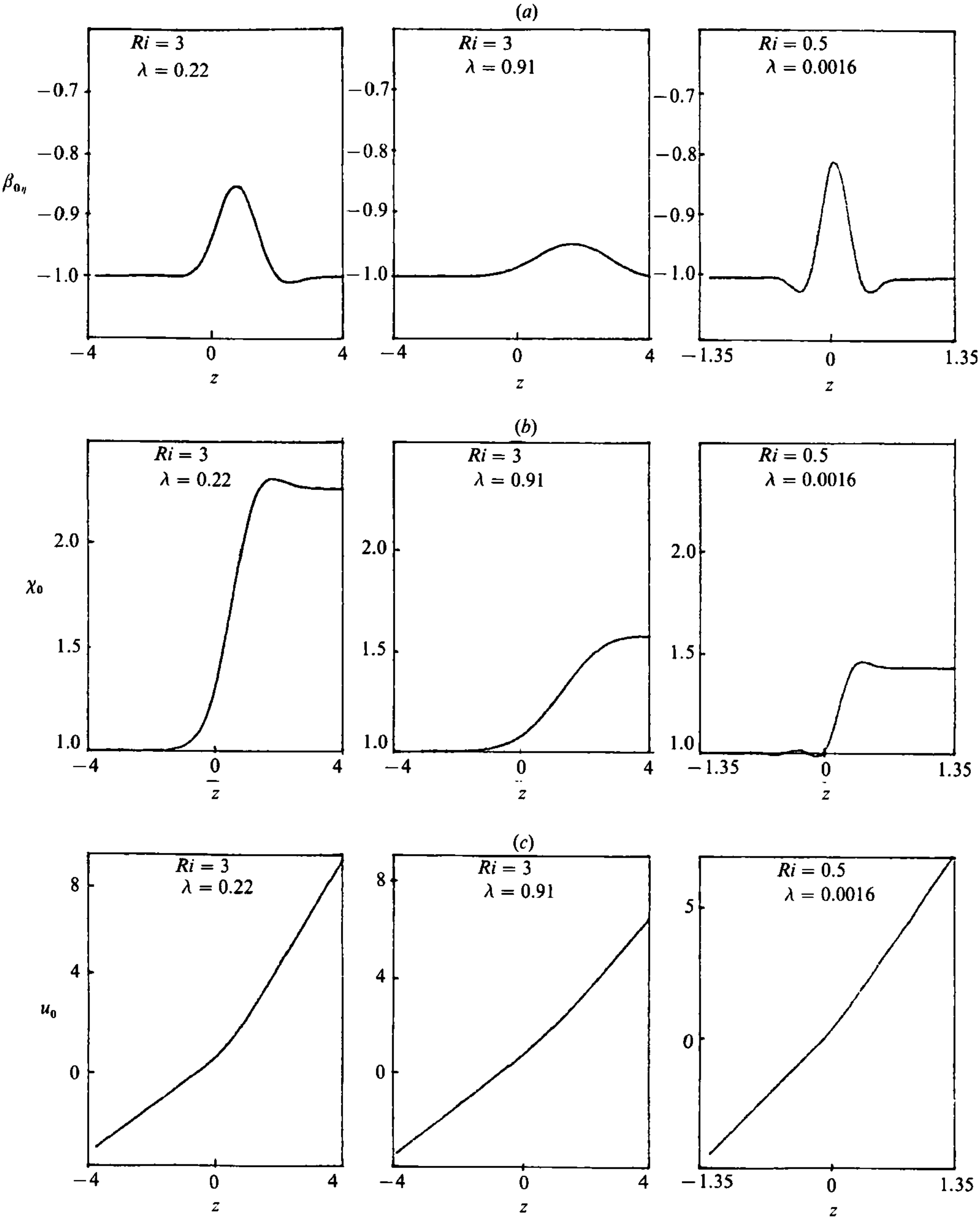

FIGURE 13. Profiles, averaged over a wave period, of (*a*) density gradient $b_{\eta 0}$, (*b*) vorticity χ_0, (*c*) velocity u_0; $Pr = 0.71$.

balanced by a viscous force. As a result a flow is formed in which the asymptotic values of the vorticity far from the CL are constant but different. No jump in the Brunt–Väisälä frequency across the CL takes place because there is no vertical mass flux in the internal waves either. Thus the Richardson numbers above and below the CL are different. In addition, the jumps in the average velocity and density across the CL occur at order $\epsilon^{\frac{2}{9}}$. The value of the vorticity jump depends on the parameter λ (the inverse vertical Reynolds number in the vicinity of the CL), it grows as λ tends to zero or as the wave amplitude grows. A wave propagating toward the CL is reflected and partially transmitted. The relationships between the vorticity jump and the wave reflection and transmission coefficients are constructed for a set of Richardson numbers.

An interesting feature of the flow occurs when λ tends to zero. In this case the value of the velocity shear on the incident-wave side tends to the value for which $Ri = \frac{1}{4}$, the reflection coefficient tends to minus unity and the transmission coefficient tends to zero. And if the incident wave amplitude increases (or λ decreases) the reflection coefficient becomes closer to minus unity, so the total wave field in the vicinity of the CL (incident one plus reflected one) does not depend on the incident wave amplitude. The total wave field is considerably smaller than the incident one, i.e. the CL is really less 'nonlinear' than would follow from the amplitude of the incident wave, and the structure of the flow in the CL is independent of the incident wave amplitude, and the Reynolds number in the CL vicinity remains fixed and not large when $\lambda \rightarrow 0$. Thus, for $Ri = 0.5$, $Re_{new} \approx 10$; for $Ri = 1$, $Re_{new} \approx 30$; for $Ri = 2$, $Re_{new} \approx 60$; for $Ri = 3$, $Re_{new} \approx 70$. As a result the marginally sufficient number of harmonics required for the solution is not very large (less than 10).

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